MA1521 Calculus for Computing

Limits

- **Maxima**: $f(c)$ is local maximum if $f(c) \geq f(x)$ for x near c (note that inequality is non-strict)
- **Differentiability**: f is differentiable at a if $f'(a)$ exists
  f is differentiable (in its domain) if f is differentiable at every point in its domain
  f is differentiable at point $a \implies f$ is continuous at a (but not the converse)
- **Critical Point**: c is a critical point if c is an interior point in the domain and either $f'(c) = 0$ or $f'(c)$ does not exist (A critical point may not be a local extremum)
- **Inflection Point**: c is an inflection point if concavity changes at c
  (c is an inflection point $\implies f''(c) = 0$, but the converse is not always true)
- $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$
- **L'Hôpital's Rule**:
  $f(a) = g(a) = 0$ or $\infty \implies \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$
  use $\ln(\dots)$ for other forms: $\infty - \infty$, 1^∞, ∞^0, 0^0

Differentiation & Integration

- **Common derivatives**:

$$
\begin{aligned}
y &\longrightarrow \frac{dy}{dx} \\
\sin x &\longrightarrow \cos x \\
\cos x &\longrightarrow -\sin x \\
\tan x &\longrightarrow \sec^2 x \\
\cot x &\longrightarrow -\csc^2 x \\
\sec x &\longrightarrow \sec x \tan x \\
\csc x &\longrightarrow -\csc x \cot x \\
a^x &\longrightarrow a^x \ln a
\end{aligned}
$$

$$
\begin{aligned}
\log_a x &\longrightarrow \tfrac{1}{x \ln a} \\
\sin^{-1} x &\longrightarrow \tfrac{1}{\sqrt{1-x^2}} \\
\cos^{-1} x &\longrightarrow -\tfrac{1}{\sqrt{1-x^2}} \\
\tan^{-1} x &\longrightarrow \tfrac{1}{1+x^2} \\
\cot^{-1} x &\longrightarrow -\tfrac{1}{1+x^2} \\
\sec^{-1} x &\longrightarrow \tfrac{1}{|x|\sqrt{x^2-1}} \\
\csc^{-1} x &\longrightarrow -\tfrac{1}{|x|\sqrt{x^2-1}} \\
x^x &\longrightarrow x^x(\ln x + 1)
\end{aligned}
$$

- **Common integrals**:

$$
\begin{aligned}
y &\longrightarrow \int y\,dx \\
\sin x &\longrightarrow -\cos x + C \\
\cos x &\longrightarrow \sin x + C \\
\tan x &\longrightarrow -\ln|\cos x| + C \\
\cot x &\longrightarrow \ln|\sin x| + C \\
\sec x &\longrightarrow \ln|\sec x + \tan x| + C \\
\csc x &\longrightarrow -\ln|\csc x + \cot x| + C
\end{aligned}
$$

- **Trigonometric formulae**: $1 + \cot^2 x = \csc^2 x$
- **Power rule**: $\frac{d}{dx} x^n = nx^{n-1}$
- **Product rule**: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- **Quotient rule**: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
- **Chain rule**: $(f \circ g)'(x) = f'(g(x))g'(x)$
- **Parametric differentiation**: $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$
- **Implicit differentiation**: Differentiate both sides of an equation containing x and y, then solve the resulting equation for the $\frac{dy}{dx}$ term.
- **Integration by parts**: $\int u\,dv = uv - \int v\,du$
  Try to differentiate in this order (highest to lowest priority):

  $\ln x \quad x^n \quad e^x, e^{-x} \quad \sin x, \cos x$

- $\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$
- $\frac{d}{dx} \int_a^{g(x)} f(t)\ dt = f(g(x))\, g'(x)$
- Integral with $\sqrt{R^2 - x^2}$: Sub $x = R\sin\theta$ or $x = R\cos\theta$
- Integral with $\sqrt{R^2 + x^2}$: Sub $x = R\tan\theta$

- **Notable formula**:

$$
\begin{aligned}
&\int \frac{A\cos\theta + B\sin\theta}{\cos\theta + \sin\theta} d\theta \\
&= \int \frac{\frac{A+B}{2}(\cos\theta + \sin\theta) + \frac{A-B}{2}(\cos\theta - \sin\theta)}{\cos\theta + \sin\theta} d\theta \\
&= \int \frac{A+B}{2} d\theta + \int \frac{A-B}{2} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} d\theta \\
&= \frac{A+B}{2}\theta + \frac{A-B}{2} \ln(\cos\theta + \sin\theta) + C
\end{aligned}
$$

- **Trigonometric formulae**:

Pythagorean

$$
\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\tan^2 x + 1 &= \sec^2 x \\
\cot^2 x + 1 &= \csc^2 x
\end{aligned}
$$

Triple angle

$$
\begin{aligned}
\sin 3x &= 3\sin x - 4\sin^3 x \\
\cos 3x &= 4\cos^3 x - 3\cos x
\end{aligned}
$$

Double angle

$$
\begin{aligned}
\sin 2x &= 2\sin x \cos x \\
\cos 2x &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 \\
&= 1 - 2\sin^2 x \\
\tan 2x &= \frac{2\tan x}{1 - \tan^2 x} \\
\sinh 2x &= 2\sinh x \cosh x \\
\cosh 2x &= 2\cosh^2 x - 1 \\
\tanh 2x &= \frac{2\tanh x}{1 + \tanh^2 x}
\end{aligned}
$$

Sum of angles

$$
\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\tan(\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \\
\cot(\alpha \pm \beta) &= \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}
\end{aligned}
$$

Linear combination of sine and cosine

$$a\sin\theta + b\cos\theta = \sqrt{a^2 + b^2}\sin(\theta + \operatorname{arctan2}(b, a))$$

Series

- **Convergence**: A sequence $\{a_n\}$ is convergent if for some fixed $L \in \mathbb{R}$, we have $\lim_{n\to\infty} a_n = L$
  Otherwise, the sequence is divergent

- **Partial sum of geometric series**: $s_n = a\dfrac{1-r^n}{1-r}$, $r \neq 1$
  s_n converges $\iff |r| < 1$

- **Ratio test for convergence of series**:
  $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho,\ \{s_n\} = \left\{\sum_{i=1}^{n} a_i\right\} \begin{cases}\text{converges} & \text{if } \rho < 1\\ \text{diverges} & \text{if } \rho > 1\\ \text{unknown} & \text{if } \rho = 1\end{cases}$$

- **Radius of convergence**:
  The value of r, where $\rho < 1 \implies |x-a| < r$
  (for power series — may be 0, *some positive real*, or ∞)

- ***p*-series**: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}\text{converges} & \text{if } p > 1\\ \text{diverges} & \text{if } 0 \le p \le 1\end{cases}$

- **Power series about $x = a$**: $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$

- **Differentiation & integration of power series**:
  $$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$
  radius of convergence is unchanged by differentiation
  $$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$
  radius of convergence is unchanged by integration

- **Taylor series**: $f(x) = \displaystyle\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$

- **Maclaurin series**: Taylor series with $a = 0$

- **Common Maclaurin series**:
  for $-1 < x < 1$
  $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$
  $$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$
  $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$$
  $$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
  $$\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$
  $$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$
  $$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{1}{2}(2 + 6x + 12x^2 + \cdots)$$
  $$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n = 1 - 2x + 3x^2 - 4x^3 + \cdots$$
  for $-\infty < x < +\infty$
  $$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
  $$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
  $$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- **Deriving more Maclaurin series**:
  Substitution: $f(x) = g(x) \implies f(h(x)) = g(h(x))$
  Multiplication: $f(x) = g(x) \implies h(x)f(x) = h(x)g(x)$
  Differentiation: $f(x) = g(x) \implies f'(x) = g'(x)$
  Integration: $f(x) = g(x) \implies \int_0^x f(x)dx = \int_0^x g(x)dx$
  ... and other usual operations on functions

- **Taylor polynomial**:
  n^{th} order Taylor polynomial: $P_n(x) :=$ terms until (and including) x^n, use for approximation
  remainder of order n: $R_n(x) :=$ remaining terms
  error := absolute value of remaining terms = $|R_n(x)|$

- **Taylor's Theorem**: $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$
  where $x \le c \le a$ (when $x \le a$) or $a \le c \le x$ (when $a \le x$)
  This provides an upper bound for the error term

Partial Differentiation

- Partial derivative wrt x – denoted by $f_x(a,b)$ or $\left.\dfrac{\partial f}{\partial x}\right|_{(a,b)}$

- $f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}$

- $f_{xy} = f_{yx}$ (when f is continuous in the neighbourhood)

- To check if $f(x,y)$ has partial derivatives of all orders, check if $f_{xy} = f_{yx}$

- **Chain rule**: For $z = f(x,y)$ and $x = x(t)$, $y = y(t)$:
  $$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
  For $z = f(x,y)$ and $x = x(s,t)$, $y = y(s,t)$:
  $$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

- **Directional derivative**: For unit vector $u = u_1\mathbf{i} + u_2\mathbf{j}$:
  $$D_u f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

- **Gradient vector**: $\nabla f(a,b) = f_x(a,b)\mathbf{i} + f_y(a,b)\mathbf{j}$
  $\nabla f(a,b) \cdot u = D_u f(a,b)$
  Direction of $\nabla f(a,b)$ is the steepest direction, maximum value of $D_u f(x,y)$ is $\|\nabla f(x,y)\| = \sqrt{f_x(a,b)^2 + f_y(a,b)^2}$

- **Maxima**: $f(a,b)$ is local maximum if $f(a,b) \ge f(x_1,y_1)$ for all points (x_1,y_2) near (a,b) (note that inequality is non-strict)

- **Critical point**: (a,b) is a critical point if $f_x(a,b) = 0$ and $f_y(a,b) = 0$ (not all critical pts are min/max points)
  A maximum point might not be a critical point if f is not smooth

- **Discriminant**: $D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$
  If $D > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum
  If $D > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum
  If $D < 0$ then (a,b) is a saddle point
  If $D = 0$ then no conclusion can be drawn
  For finding max/min, f_{yy} may be used in place of f_{xx}

Ordinary Differential Equations

- **Definition**: $\sum_{i=0}^{n} a_i(x)y^{(i)}(x) = F(x)$
  where $a_i(x)$ and $F(x)$ are functions of x
  and $y^{(i)}(x)$ is the i^{th} derivative of y w.r.t. x

- **Separable equations**: Separate and integrate both sides

- **Exponential decay**: $\frac{dx}{dt} = kx \implies x(t) = x(0)e^{kt}$
  $x(0)$ is the initial val., $k = -\frac{\ln 2}{\tau}$ where τ is the half-life

- **Exponential cooling/heating**:
  $\frac{dx}{dt} = k(x - x_0) \implies x(t) - x_0 = (x(0) - x_0)\, e^{kt}$
  $x(0)$ is the initial value, x_0 is the target value

- **Hyperbolic functions**: $\sinh(x) = \frac{e^x - e^{-x}}{2}$
  $\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

- **Reduction to separable form**: If ODE contains fraction where the degree of terms are equal, then we can substitute:
  $\frac{dy}{dx} = \frac{y^2 - x^2}{xy} = \frac{y}{x} - \frac{x}{y} \implies x\frac{dv}{dx} + v = v - \frac{1}{v}$
  where $v = \frac{y}{x}$

- **Substitution**: Substitute $u = f(x, y)$ to expr. and $\frac{dy}{dx}$

- **Linear 1st order ODEs**: $\frac{dy}{dx} + p(x)y = q(x)$
  $\implies yf(x) = \int q(x)f(x)dx$ where $f(x) := e^{\int p(x)dx}$

- **Bernoulli equation**: $\frac{dy}{dx} + p(x)y = q(x)y^n$
  $\implies \frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$ where $z = y^{1-n}$
  ... which is a linear 1st order ODE

Homogeneous linear 2nd order ODEs

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

- **Superposition principle**: If y_1 and y_2 are solutions then $c_1y_1 + c_2y_2$ is also a solution (i.e. the solution set is a subspace)

- **Dimension**: The solution space has dimension 2, so finding 2 linearly independent solutions is sufficient to obtain the general solution

- **Guessing**: Obtain 2 lin. indep. solutions by guessing

- **Constant $p(x)$ and $q(x)$**:
  Let $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$, then solutions have form $y = e^{\lambda x}$ for some value λ
  By substitution, $\lambda^2 + A\lambda + B = 0$, solve for $\lambda = \lambda_1, \lambda_2$
  Two distinct real roots: $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
  Two repeated (real) roots: $y = c_1e^{\lambda x} + c_2xe^{\lambda x}$
  Two distinct complex roots: If λ_1 or $\lambda_2 = a + b\sqrt{-1}$
  then $y = e^{ax}(c_1\cos bx + c_2\sin bx)$
  Given λ_1, λ_2 we can recover $A = -(\lambda_1 + \lambda_2), B = \lambda_1\lambda_2$

Non-homogeneous linear 2nd order ODEs

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)$$

Find a particular solution y_p for this equation, and find the general solution y_h for the homogeneous version ($f(x) = 0$), then the general solution for this equation is $y = y_h + y_p$

Mathematical Modelling

- **Malthusian growth model**:
  $B :=$ per capita birth rate (constant)
  $D :=$ per capita death rate (constant)
  $\frac{dN}{dt} = (B - D)N \implies N(t) = N(0)e^{(B-D)t}$
  $B > D \implies$ population explosion
  $B = D \implies$ stable
  $B < D \implies$ extinction

- **Logistic growth model**:
  $B :=$ per capita birth rate (constant)
  $sN :=$ per capita death rate (linear to population)
  $\frac{dN}{dt} = (B - sN)N \implies$
  $N(t) = \frac{B}{s + (\frac{B}{N(0)} - s)e^{-Bt}} = \frac{\frac{B}{s}}{1 + \left(\frac{B}{s} \cdot \frac{1}{N(0)} - 1\right)e^{-Bt}}$
  $B - sN(t) > 0 \quad \forall t \implies \frac{B}{s} > N(t)$
  $\implies$ smaller than sustainable population
  $B - sN(t) = 0 \quad \forall t \implies \frac{B}{s} = N(t)$
  $\implies$ sustainable population (equilibrium)
  $B - sN(t) < 0 \quad \forall t \implies \frac{B}{s} < N(t)$
  $\implies$ larger than sustainable population
  Population always tends to $\boxed{\frac{B}{s}}$ — "carrying capacity"

- **Harvesting growth model**:
  $B :=$ per capita birth rate (constant)
  $sN :=$ per capita death rate (linear to population)
  $E :=$ harvest rate
  $\frac{dN}{dt} = (B - sN)N - E = BN - sN^2 - E$
  Solve $-sN^2 + BN - E = 0$ for equilibrium solutions
  $B^2 - 4sE > 0 \implies \frac{B^2}{4s} > E \implies$ two equilibriums
  $\implies \beta_1 < \beta_2 < \frac{B}{s}$ and $\beta_1 + \beta_2 = \frac{B}{s}$
  β_2 is stable but not β_1
  $B^2 - 4sE = 0 \implies \frac{B^2}{4s} = E \implies$ one equilibrium
  $\implies \beta = \frac{B}{2s}$
  $B^2 - 4sE < 0 \implies \frac{B^2}{4s} < E \implies$ no equilibrium

Useful Sequences

Prime Numbers

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	···

$1521 = 3^2 \times 13^2$
2017 is prime